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Relative differential K-characters

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Abstract. We define a group of relative differential K-characters associated with a smooth map between two smooth compact manifolds. We show that this group fits into a short exact sequence as in the non-relative case. Some secondary geometric invariants are expressed in this theory.

Key words: geometric K-homology; differential K-characters

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1 Introduction

Cheeger and Simons [10] introduced the notion of differential characters to express some secondary geometric invariants of a principal G-bundle in the base space. This theory has been appearing more and more frequently in quantum field and string theories (see [7, 15, 13]). On the other hand, it was shown recently (see [4, 16, 17]) that K-homology of Baum–Douglas [5] is an appropriate arena in which various aspects of D-branes in superstring theory can be described.

In [8] we have defined with M.T. Benameur the notion of differential characters in K-theory on a smooth compact manifold. Our original motivation was to explain some secondary geometric invariants coming from the Chern–Weil and Cheeger–Simons theory in the language of K-theory. To do this, we have used the Baum–Douglas construction of K-homology. As a result, we obtained the eta invariant of Atiyah–Patodi–Singer as a $\mathbb{R}/\mathbb{Z}$-differential K-character, while it is a $\mathbb{R}/\mathbb{Q}$-invariant in the works of Cheeger and Simons. Recall that a geometric K-cycle of Baum–Douglas over a smooth compact manifold X is a triple (M, E, ϕ) such that: M is a closed smooth Spin^c-compact manifold with a fixed Riemannian structure; E is a Hermitian vector bundle over M with a fixed Hermitian connection ∇^E and $\phi : M \to X$ is a smooth map. Let $\mathcal{C}_*(X)$ be the semi-group for the disjoint union of equivalence classes of K-cycles over X generated by direct sum and vector bundle modification [5]. A differential K-character on X is a homomorphism of semi-group $\varphi : \mathcal{C}_*(X) \to \mathbb{R}/\mathbb{Z}$ such that it is restriction to the boundary is given by the formula

$$\varphi(\partial(M, E, \phi)) = \int_M \phi^*(\omega)\mathrm{Ch}(E)\mathrm{Td}(M) \mod \mathbb{Z},$$

where ω is a closed form on X with integer K-periods [8], $\mathrm{Ch}(E)$ is the Chern form of the connection ∇^E and $\mathrm{Td}(M)$ is the Todd form of the tangent bundle of M. This can be assembled into a group which is denoted by $\hat{K}^*(X)$ and called the group of differential K-characters. We showed then that many secondary invariants can be expressed as a differential K-characters, and the group $K^*(X, \mathbb{R}/\mathbb{Z})$ of K-theory of X with coefficients in $\mathbb{R}/\mathbb{Z}$ [2] is injected in $\hat{K}^*(X)$.

The aim of this work is to define the group $\hat{K}^*(\rho)$ of relative differential K-characters associated with a smooth map $\rho : Y \to X$ between two smooth compact manifolds Y and X following [9, 12] and [13]. We show that this group fits into a short exact sequence as in the non-relative case. The paper is organized as follows:

In Section 2, we define a group of relative geometric K-homology $K_*(\rho)$ adapted to this situation and study some of its properties. This generalizes the works of Baum–Douglas [6] for Y a submanifold of X. Section 3 is concerned with the definition and the study of the group $\hat{K}^*(\rho)$ of relative differential K-characters. An odd relative group $K^{-1}(\rho,\mathbb{R}/\mathbb{Z})$ of K-theory with coefficients in $\mathbb{R}/\mathbb{Z}$ is also defined here. We proof the following short exact sequence

$$0 \to K^{-1}(\rho,\mathbb{R}/\mathbb{Z}) \hookrightarrow \hat{K}^{-1}(\rho) \xrightarrow{\delta_1} \Omega_0^{\mathrm{even}}(\rho) \to 0,$$

where $\Omega_0^{\mathrm{even}}(\rho)$ is the group of relative differential forms (Definition 6) with integer K-periods. We show then that some secondary geometric invariants can be expressed in this theory.

2 Relative geometric K-homology

Let Y and X be smooth compact manifolds and $\rho : Y \to X$ a smooth map. In this section, we define the relative geometric K-homology $K_*(\rho)$ for the triple (ρ, Y, X). This construction generalizes the relative geometric K-homology group $K_*(X,Y)$ of Baum–Douglas for Y being a closed submanifold of X. We recall the definition of the geometric K-homology of a smooth manifold following the works of Baum and Douglas. This definition is purely geometric. For a complete presentation see [5, 6] and [17].

Definition 1. A K-chain over X is a triple (M,E,ϕ) such that:

- M is a smooth Spin^c-compact manifold which may have non-empty boundary ∂M, and with a fixed Riemannian structure;
- E is a Hermitian vector bundle over M with a fixed Hermitian connection ∇^E;
- $\phi : M \to X$ is a smooth map.

Denote that M is not supposed connected and the fibres of E may have different dimensions on the different connected components of M. Two K-chains (M,E,ϕ) and (M',E',ϕ') are said to be isomorphic if there exists a diffeomorphism $\psi : M \to M'$ such that:

- $\phi' \circ \psi = \phi$;
- $\psi^* E' \cong E$ as Hermitian bundles over M.

A K-cycle is a K-chain (M,E,φ) without boundary; that is $\partial M = \varnothing$. The boundary $\partial(M,E,\varphi)$ of the K-chain (M,E,φ) is the K-cycle $(\partial M, E|_{\partial M}, \varphi|_{\partial M})$. The set of K-chains is stable under disjoint union.

2.1 Vector bundle modification

Let (M,E,ϕ) be K-chain over X, and let H be a Spin^c-vector bundle over M with even dimensional fibers and a fixed Hermitian structure. Let $l = M \times \mathbb{R}$ be the trivial bundle and $\hat{M} = S(H \oplus l)$ the unit sphere bundle. Let $\rho : \hat{M} \to M$ the natural projection. The Spin^c-structure on M and H induces a Spin^c-structure on $\hat{M}$.

Let $\mathcal{S} = \mathcal{S}_- \oplus \mathcal{S}_+$ be the $\mathbb{Z}/2\mathbb{Z}$-grading Clifford module associated with the Spin^c-structure of H. We denote by H_0 and H_1 the pullback of $\mathcal{S}_-$ and $\mathcal{S}_+$ to H. Then H acts on H_0 and H_1 by Clifford multiplication: $H_0 \xrightarrow{\sigma} H_1$.

The manifold $\hat{M}$ can be thought of as two copies, $B_0(H)$ and $B_1(H)$, of the unit ball glued together by the identity map of $S(H)$

$$\hat{M} = B_0(H) \cup_{S(H)} B_1(H).$$

The vector bundle $\hat{H}$ on $\hat{M}$ is obtained by putting H_0 on $B_0(H)$ and H_1 on $B_1(H)$ and then clutching these two vector bundles along $S(H)$ by the isomorphism σ.

The K-chain $(\hat{M}, \hat{H} \otimes \rho^* E, \hat{\phi} = \rho \circ \phi)$ is called the Bott K-chain associated with the K-chain (M, E, ϕ) and the Spin^c-vector bundle H.

The boundary of the Bott K-chain $(\hat{M}, \hat{H} \otimes \rho^* E, \hat{\phi})$ associated with the K-chain (M, E, ϕ) and the Spin^c-vector bundle H is the Bott K-cycle of the boundary $\partial(M, E, \phi)$ with the restriction of H to ∂M.

Definition 2. We denote by $\mathcal{C}_*(X)$ the set of equivalence classes of isomorphic K-cycles over X up to the following identifications:

- we identify the disjoint union $(M, E, \phi) \amalg (M, E', \phi)$ with the K-cycle $(M, E \oplus E', \phi)$;
- we identify a K-cycle (M, E, ϕ) with the Bott K-cycle $(\hat{M}, \hat{H} \otimes \rho^* E, \hat{\phi})$ associated with any Hermitian vector bundle H over M.

We can easily show that disjoint union then respects these identifications and makes $\mathcal{C}_*(X)$ into an Abelian semi-group which splits into $\mathcal{C}_0(X) \oplus \mathcal{C}_1(X)$ with respect to the parity of the connected components of the manifolds in (the equivalence classes of) the K-cycles.

Definition 3. Two K-cycles (M, E, ϕ) and (M', E', ϕ') are bordant if there exists a K-chain $(\overline{N}, \mathcal{E}, \psi)$ such that

$$\partial(\overline{N}, \mathcal{E}, \psi) \quad \text{is isomorphic to} \quad (M, E, \phi) \amalg (-M', E', \phi'),$$

where $-M'$ is M' with the Spin^c-structure reversed [5] .

The above bordism relation induces a well defined equivalence relation on $\mathcal{C}_*(X)$ that we denote by $\sim_\partial$. The quotient $\mathcal{C}_*(X)/\sim_\partial$ turns out to be an Abelian group for the disjoint union. The inverse of (M, E, ϕ) is $(-M, E, \phi)$.

Definition 4 (Baum–Douglas)**.** The quotient group of $\mathcal{C}_*(X)$ by the equivalence relation $\sim_\partial$ is denoted by $K_*(X)$ and is called the geometric K-homology group of X. It can be decomposed into

$$K_*(X) = K_0(X) \oplus K_1(X).$$

A smooth map $\varphi : Y \to X$ induces a group morphism

$$\varphi_* : K_*(Y) \to K_*(X),$$

given by $\varphi_*(f)(M, E, \phi) = f(M, E, \varphi \circ \phi)$. The K_* is a covariant functor from the category of smooth compact manifolds and smooth maps to that of Abelian groups and group homomorphisms.

In the same way we can form a semi-group $\mathcal{L}_*(X)$ out of K-chains $(\overline{N}, \mathcal{E}, \psi)$, say with the same definition as $\mathcal{C}_*(X)$ and the *boundary*

$$\partial(\overline{N}, \mathcal{E}, \psi) = (\partial\overline{N}, \mathcal{E}|_{\partial\overline{N}}, \psi \circ i),$$

where $i : \partial\overline{N} \hookrightarrow \overline{N}$. This gives a well defined map

$$\partial : \mathcal{L}_*(X) \to \mathcal{C}_*(X) \subset \mathcal{L}_*(X).$$

The Hermitian structure of the complex vector bundle $\mathcal{E}|_{\partial\overline{N}}$ is the restricted one.

The group of K-cochains with coefficients in $\mathbb{Z}$ denoted by $\mathcal{L}^*(X)$ is the group of semi-group homomorphisms f from $\mathcal{L}_*(X)$ to $\mathbb{Z}$. On the group $\mathcal{L}^*(X)$ there is a coboundary map defined by transposition

$$\delta(f)(\overline{N}, \mathcal{E}, \psi) = f(\partial(\overline{N}, \mathcal{E}, \psi)).$$

The set of K-cocycles is the subset $\mathcal{C}^*(X)$ of $\mathcal{L}^*(X)$ of those K-cochains that vanish on boundaries, i.e. the kernel of δ. The set of K-coboundaries is the image of δ in $\mathcal{L}^*(X)$.

2.2 The relative geometric group $K_*(\rho)$

Let Y and X be smooth compact manifolds and $\rho : Y \to X$ a smooth map.

The set $\mathcal{L}_*(\rho)$ of relative K-chains associated with the triple (ρ, Y, X) is by definition

$$\mathcal{L}_{*+1}(\rho) = \mathcal{L}_{*+1}(X) \times \mathcal{L}_*(Y).$$

The boundary $\partial : \mathcal{L}_{*+1}(\rho) \to \mathcal{L}_*(\rho)$ is given by

$$\partial(\sigma, \tau) = (\partial\sigma + \rho_*\tau, -\partial\tau).$$

We will denote by $\mathcal{C}_*(\rho)$ the set of relative K-cycles in $\mathcal{L}_*(\rho)$, i.e., the kernel of ∂. A K-cycle in $\mathcal{L}_*(\rho)$ is then a pair (σ, τ) where τ is a K-cycle over Y and σ is K-chain over X with boundary in the image of $\rho_* : \mathcal{C}_*(Y) \to \mathcal{C}_*(X)$. The set $\mathcal{C}_*(\rho)$ is a semi-group for the sum

$$(\sigma, \tau) + (\sigma', \tau') = (\sigma \amalg \sigma', \tau \amalg \tau'),$$

where $\amalg$ is the disjoint union. We say that two relatives K-cycles(σ, τ) and (σ', τ') are bordant and we write $(\sigma, \tau) \sim_\partial (\sigma', \tau')$ if there exists a relative K-chain $(\overline{\sigma}, \overline{\tau})$ such that

$$\partial(\overline{\sigma}, \overline{\tau}) = (\sigma, \tau) + (-\sigma', -\tau'),$$

where $-x$ denotes the relative K-cycle x with the reversed Spin^c-structure of the underlying manifold.

Definition 5. The relative geometric K-homology group denoted by $K_*(\rho)$ is the quotient group $\mathcal{C}_*(\rho)/\sim_\partial$.

The inverse of the K-cycle x is $-x$. The equivalence relation on the relative K-cycle (σ, τ) preserves the dimension modulo 2 of the K-cycles σ and τ. Hence, there is a direct sum decomposition

$$K_*(\rho) = K_0(\rho) \oplus K_1(\rho).$$

The construction of the group $K_*(\rho)$ is functorial in the sense that for a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\rho"] \arrow[d, "f"] & X \arrow[d, "g"] \\ Y' \arrow[r, "\rho'"] & X' \end{tikzcd}$$

the map $F_* = (f_*, g_*) : \mathcal{L}_*(\rho) \to \mathcal{L}_*(\rho')$ is compatible with the equivalence relation on the relative K-cycles and induces a homomorphism from $K_*(\rho)$ to $K_*(\rho')$. As in the homology theory, we have the long exact sequence for the triple (ρ, Y, X)

$$\begin{tikzcd} K_0(Y) \arrow[r, "\rho_*"] & K_0(X) \arrow[r, "\varsigma_*"] & K_0(\rho) \arrow[d, "\partial"] \\ K_1(\rho) \arrow[u, "\partial"] & K_1(X) \arrow[l, "\varsigma_*"] & K_1(Y) \arrow[l, "\rho_*"] \end{tikzcd}$$

The boundary map ∂ associates to a relative K-cycle (σ, τ) the cycle τ whose image $\rho_*\tau$ is a boundary in X and $\varsigma_*(\sigma) = (\sigma, 0)$. The exactness of the diagram is an easy check.

There is a differential δ on the group $\mathcal{L}^*(\rho) = \mathrm{Hom}(\mathcal{L}_*(\rho), \mathbb{Z})$ given by

$$\delta(h, e) = (\delta h, \rho^* h - \delta e).$$

The relative Baum–Douglas K-group is

$$K^*(\rho) = \frac{\ker(\delta : \mathcal{L}^*(\rho) \to \mathcal{L}^{*+1}(\rho))}{\mathrm{Im}(\delta : \mathcal{L}^{*-1}(\rho) \to \mathcal{L}^*(\rho))}.$$

Remark 1. The relative topological K-homology group $K^t_*(\rho)$ can be constructed in the same way for normal topological spaces X and Y, and $\rho : Y \to X$ is a continuous map. Let $K^t_*(X, Y)$ be the relative topological K-homology group defined by Baum–Douglas in [6] for $Y \subset X$ is a closed subset of a X. We can easily show that $K^t_*(X, Y) = K^t_*(\rho)$, where ρ is the inclusion of Y in X.

3 Relative differential K-characters

This section is concerned with the definition and the study the notion of relative differential K-characters [8]. This is a K-theoretical version of the works of [9, 12] and [13].

Let X be a smooth compact manifold. The graded differential complex of real differential forms on the manifold X will be denoted by

$$\Omega^*(X) = \oplus_{k\geq 0}\Omega^k(X), \qquad \Omega^k(X) \xrightarrow{d} \Omega^{k+1}(X) \qquad \text{with} \quad d^2 = 0,$$

where d denotes the de Rham differential on X.

Furthermore, we denote by $\Omega^*_0(X)$ the subgroup of closed forms on the manifold X with integer K-periods [8].

In the remainder of this section we fix $\rho : Y \to X$ a smooth map and we consider the complex

$$\Omega^*(\rho) = \Omega^*(X) \times \Omega^{*-1}(Y)$$

with differential $\delta(\omega, \theta) = (d\omega, \rho^*\omega - d\theta)$.

We can view $\Omega^*(\rho)$ as a subgroup of the the group $\mathrm{Hom}(\mathcal{L}_*(\rho), \mathbb{R})$ via integration

$$(\omega, \theta)(\sigma, \tau) = \omega(\sigma) + \theta(\tau),$$

where for $\sigma = (M, E, f)$ and $\tau = (N, F, g)$

$$\omega(\sigma) = \int_M f^*(\omega)\mathrm{Ch}(E)\mathrm{Td}(M) \qquad \text{and} \qquad \theta(\tau) = \int_N g^*(\theta)\mathrm{Ch}(F)\mathrm{Td}(N).$$

Let

$$j : \Omega^*(\rho) \to \mathrm{Hom}(\mathcal{L}_*(\rho), \mathbb{R})$$

such that

$$j(\omega, \theta)(\sigma, \tau) = \omega(\sigma) + \theta(\tau).$$

Definition 6. Let $(\omega, \theta) \in \Omega^*(\rho)$ be a pair of real differential forms.

(i) The set of K-periods of (ω, θ) is the subset of $\mathbb{R}$ image of the map $j(\omega, \theta)$ restricted to $\mathcal{C}_*(\rho)$.

(ii) We denoted by $\Omega^*_0(\rho)$ the set of differential forms (ω, θ) of integer K-periods.

$\Omega^*_0(\rho)$ is an Abelian group for the sum of differential forms.

Lemma 1. *Let $(\omega, \theta) \in \Omega^*_0(\rho)$. Then*

1) *$\delta(\omega, \theta) = 0$ in the complex $\Omega^*(\rho)$;*

2) *$\omega \in \Omega^*_0(X)$.*

Proof. 1) For $(\omega,\theta)\in\Omega_0^*(\rho)$ and $\tau=(N,F,g)\in\mathcal{L}_{*-1}(Y)$, we have

$$\begin{aligned}\rho^*\omega(\tau)-d\theta(\tau) &= \int_N (g^*(\rho^*(\omega))-g^*(d\theta))\mathrm{Ch}(F)\mathrm{Td}(N)\\ &= \int_N g^*(\rho^*(\omega))\mathrm{Ch}(F)\mathrm{Td}(N)-\int_{\partial N} g^*(\theta)\mathrm{Ch}(F)\mathrm{Td}(N)\\ &= \int_N (\rho\circ g)^*(\omega)\mathrm{Ch}(F)\mathrm{Td}(N)-\int_{\partial N} g^*(\theta)\mathrm{Ch}(F)\mathrm{Td}(N)\\ &= (\omega,\theta)(\rho_*\tau,-\partial\tau).\end{aligned}$$

Since $(\omega,\theta)\in\Omega_0^*(\rho)$ and $(\rho_*\tau,-\partial\tau)=\partial(0,\tau)$ is a relative K-cycle, the value $(\omega,\theta)(\rho_*\tau,-\partial\tau)$ is entire. Lemma 3 of [8] implies that $\rho^*\omega-d\theta=0$. On the other hand, for any K-chain $\sigma\in\mathcal{L}(X)$, we have $d\omega(\sigma)=(\omega,\theta)(\partial\sigma,0)$. Since $(\partial\sigma,0)$ is a relative K-cycle, it follows for the same raison that $d\omega=0$.

2) Let $\sigma=(M,E,f)\in\mathcal{C}_*(X)$. We have

$$\int_M f^*(\omega)\mathrm{Ch}(E)\mathrm{Td}(M)=(\omega,\theta)(\sigma,0).$$

Since (ω,θ) has integer K-periods and $(\sigma,0)$ is a relative K-cycle, the right hand-side is entire. ■

Example 1. Any pair $(\omega,\theta)\in\Omega^*(\rho)$ of exact differential forms is obviously in $\Omega_0^*(\rho)$.

Remark 2. We can easily deduce from the proof of the previous lemma that an element $(\omega,\theta)\in\Omega^*(\rho)$ with entire values on all K-chains is necessarily trivial.

Definition 7.

(i) A relative differential K-character for the smooth map $\rho: Y\to X$ is a homomorphism of semi-group

$$f:\mathcal{C}_*(\rho)\to\mathbb{R}/\mathbb{Z}$$

such that $f(\partial(\sigma,\tau))=[(\omega,\theta)(\sigma,\tau)]$ for some $(\omega,\tau)\in\Omega_0^*(\rho)$ and for all relative K-chain $(\sigma,\tau)\in\mathcal{L}_*(\rho)$, where $[\alpha]$ denote the class in $\mathbb{R}/\mathbb{Z}$ of the number α.

(ii) The set of relative differential K-characters is denoted by $\hat{K}^*(\rho)$. It is naturally $\mathbb{Z}/2\mathbb{Z}$-graded

$$\hat{K}^*(\rho)=\hat{K}^0(\rho)\oplus\hat{K}^1(\rho).$$

Let f be a relative differential K-character for the smooth map $\rho: Y\to X$. We deduce from Remark 2 that the pair of forms (ω,θ) associated to f in Definition 7 is unique. It will be denoted by $\delta_1(f)$. We thus have a group morphism

$$\delta_1:\hat{K}^*(\rho)\to\Omega_0^*(\rho),$$

which is odd for the grading.

Example 2. An interesting situation is obtained by differential forms. If $(\omega,\theta)\in\Omega^*(X)\times\Omega^{*-1}(Y)$ is any pair of real differential forms, then we define $f_{(\omega,\theta)}$ by letting for $\sigma=(M,E,f)$ and $\tau=(N,F,g)$

$$f_{(\omega,\theta)}(\sigma,\tau)=\left[\int_M f^*(\omega)\mathrm{Ch}(E)\mathrm{Td}(M)\right]+\left[\int_N g^*(\theta)\mathrm{Ch}(F)\mathrm{Td}(N)\right].$$

We have

$$\delta_1(f_{(\omega,\theta)})=(d\omega,\rho^*\omega-d\theta).$$

Example 3. Suppose Y be submanifold of X and $\rho : Y \hookrightarrow X$ is a smooth inclusion. Let $\omega \in \Omega^*(X)$ with trivial restriction to Y and $\bar{f}_\omega \in \hat{K}(X)$ – the associated differential K-character [8]. Let $\psi \in \hat{K}(Y)$ be any differential K-character on Y. We have $\bar{f}_\omega(\mathcal{L}_* Y) = 0$. The map $\phi_{\omega,\psi}$ defined on $\mathcal{C}_*(\rho)$ by

$$\phi_{\omega,\psi}(\sigma, \tau) = \bar{f}_\omega(\sigma) + \psi)(\tau)$$

is a relative differential K-character with $\delta_1(\phi_{\varphi,\psi}) = (d\omega, -\delta_1(\psi))$.

3.1 Relative $\mathbb{R}/\mathbb{Z}$-K-theory

Let X be a smooth manifold, E a Hermitian vector bundle on X and ∇^E a Hermitian connection on E. The geometric Chern form $\mathrm{Ch}(\nabla^E)$ of ∇^E is the closed real even differential form given by

$$\mathrm{Ch}(\nabla^E) = \mathrm{tr}\, e^{-\frac{(\nabla^E)^2}{2i\pi}}.$$

The cohomology class of $\mathrm{Ch}(\nabla^E)$ does not depend on the choice of the connection ∇^E [14]. Let ∇_1^E and ∇_2^E be two Hermitian connections on E. There is a well defined Chern–Simons form [14] $\mathrm{CS}(\nabla_1^E, \nabla_2^E) \in \frac{\Omega^{\mathrm{odd}}(X)\otimes\mathbb{C}}{\mathrm{Im}(d)}$ such that

$$d\mathrm{CS}(\nabla_1^E, \nabla_2^E) = \mathrm{Ch}(\nabla_1^E) - \mathrm{Ch}(\nabla_2^E),$$

and

$$\mathrm{CS}(\nabla_1^E, \nabla_3^E) = \mathrm{CS}(\nabla_1^E, \nabla_2^E) + \mathrm{CS}(\nabla_2^E, \nabla_3^E).$$

Given a short exact sequence of complex Hermitian vector bundles on X

$$0 \to E_1 \xrightarrow{i} E_2 \xrightarrow{j} E_3 \to 0,$$

and choose a splitting map $s : E_3 \to E_2$. Then $i \oplus s : E_1 \oplus E_3 \to E_2$ is an isomorphism. For ∇^{E_1}, ∇^{E_2} and ∇^{E_3} are Hermitian connection on E_1, E_2 and E_3 respectively, we set

$$\mathrm{CS}(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}) = \mathrm{CS}((i \oplus s)^* \nabla^{E_2}, \nabla^{E_1} \oplus \nabla^{E_3}).$$

The form $\mathrm{CS}(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3})$ is independent of the choice of the splitting map s and

$$d\mathrm{CS}(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}) = \mathrm{Ch}(\nabla_2^E) - \mathrm{Ch}(\nabla_1^E) - \mathrm{Ch}(\nabla_3^E).$$

Definition 8. Let X be a smooth manifold. A $\mathbb{R}/\mathbb{Z}$-K-generator of X is a quadruple

$$\mathcal{E} = (E, h^E, \nabla^E, \omega),$$

where E is a complex vector bundle on X, h^E is a positive definite Hermitian metric on E, ∇^E is a Hermitian connection on E, $\omega \in \frac{\Omega^{\mathrm{odd}}(X)}{\mathrm{Im}(d)}$ which satisfies $d\omega = \mathrm{Ch}(\nabla^E) - \mathrm{rank}(E)$, where $\mathrm{rank}(E)$ is the rank of E.

An $\mathbb{R}/\mathbb{Z}$-K-relation is given by three $\mathbb{R}/\mathbb{Z}$-K-generators $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, along with a short exact sequence of Hermitian vector bundles

$$0 \to E_1 \xrightarrow{i} E_2 \xrightarrow{j} E_3 \to 0,$$

such that $\omega_2 = \omega_1 + \omega_3 + \mathrm{CS}(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3})$.

Definition 9 ([14])**.** We denote by $MK(X,\mathbb{R}/\mathbb{Z})$ the quotient of the free group generated by the $\mathbb{R}/\mathbb{Z}$-K-generators and $\mathbb{R}/\mathbb{Z}$-K-relation $\mathcal{E}_2 = \mathcal{E}_1 + \mathcal{E}_3$. The group $K^{-1}(X,\mathbb{R}/\mathbb{Z})$ is the subgroup of $MK(X,\mathbb{R}/\mathbb{Z})$ consisting of elements of virtual rank zero.

The elements of $K^{-1}(X,\mathbb{R}/\mathbb{Z})$ can be described by $\mathbb{Z}/2\mathbb{Z}$-graded cocycles [14], meaning quadruples $(E_\pm, h^{E_\pm}, \nabla^{E_\pm}, \omega)$ where $E = E_+ \oplus E_-$ is a $\mathbb{Z}/2\mathbb{Z}$-graded complex vector bundle on X, $h^E = h^{E_+} \oplus h^{E_-}$ is a Hermitian metric on E, $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$ is a Hermitian connection on E, $\omega \in \frac{\Omega^{\mathrm{odd}}(X)}{\mathrm{Im}(d)}$ and satisfies $d\omega = \mathrm{Ch}(\nabla^E) = \mathrm{Ch}(\nabla^{E_+}) - \mathrm{Ch}(\nabla^{E_-})$.

We consider now two smooth compact manifolds Y and X. Let $\rho : Y \to X$ be a smooth map and let the exact sequence

$$\begin{array}{ccccc}
K^0(Y,\mathbb{R}/\mathbb{Z}) & \overset{\rho_0^*}{\longleftarrow} & K^0(X,\mathbb{R}/\mathbb{Z}) & \overset{\varsigma_0^*}{\longleftarrow} & \mathrm{Hom}(K_0(\rho),\mathbb{R}/\mathbb{Z}) \\
\big\downarrow \partial_0^* & & & & \big\uparrow \partial_1^* \\
\mathrm{Hom}(K_1(\rho),\mathbb{R}/\mathbb{Z}) & \overset{\varsigma_1^*}{\longrightarrow} & K^{-1}(X,\mathbb{R}/\mathbb{Z}) & \overset{\rho_1^*}{\longrightarrow} & K^{-1}(Y,\mathbb{R}/\mathbb{Z})
\end{array}$$

obtained from the one in p. 4 and after identification of the groups $K^*(Y,\mathbb{R}/\mathbb{Z})$ and $\mathrm{Hom}(K_*(Y),\mathbb{R}/\mathbb{Z})$ following Proposition 4 of [8].

We denote by $\bar{L}^{-1}(\rho,\mathbb{R}/\mathbb{Z})$ the subgroup of $\mathrm{Hom}(K_1(\rho),\mathbb{R}/\mathbb{Z})$ image of the morphism ∂_0^* and $\bar{K}^{-1}(\rho,\mathbb{R}/\mathbb{Z})$ the subgroup of $K^{-1}(X,\mathbb{R}/\mathbb{Z})$ the kernel of the morphism ρ_1^*.

Definition 10. Let Y and X be two smooth compact manifolds and $\rho : Y \to X$ a smooth map. The group $K^{-1}(\rho,\mathbb{R}/\mathbb{Z})$ is by definition the product of the groups $\bar{L}^{-1}(\rho,\mathbb{R}/\mathbb{Z})$ and $\bar{K}^{-1}(\rho,\mathbb{R}/\mathbb{Z})$

$$K^{-1}(\rho,\mathbb{R}/\mathbb{Z}) = \bar{L}^{-1}(\rho,\mathbb{R}/\mathbb{Z}) \times \bar{K}^{-1}(\rho,\mathbb{R}/\mathbb{Z}).$$

Proposition 1. *The groups* $K^{-1}(\rho,\mathbb{R}/\mathbb{Z})$ *and* $\mathrm{Hom}(K_{-1}(\rho),\mathbb{R}/\mathbb{Z})$ *are isomorphic.*

Proof. Since the image of ς_1^* is the kernel of ρ_1^*, it is enough to show that the short exact sequence

$$0 \to \bar{L}^{-1}(\rho,\mathbb{R}/\mathbb{Z}) \hookrightarrow \mathrm{Hom}(K_1(\rho),\mathbb{R}/\mathbb{Z}) \overset{\varsigma_1^*}{\to} \bar{K}^{-1}(\rho,\mathbb{R}/\mathbb{Z}) \to 0$$

is split. Let $\mathcal{E}$ be an element of $\bar{K}^{-1}(\rho,\mathbb{R}/\mathbb{Z})$ and $(E_\pm, h^{E_\pm}, \nabla^{E_\pm}, \omega)$ be a relative $\mathbb{Z}/2\mathbb{Z}$-graded cocycle associated to $\mathcal{E}$. Let (σ,τ) be a relative K-cycle in $\mathcal{C}_{-1}(\rho)$. For $\sigma = (M,E,\phi)$ and $\tau = (N,F,\psi)$ we set

$$\alpha(\mathcal{E})(\sigma,\tau)) = \bar{\eta}_{\phi^*E_+\otimes E} - \bar{\eta}_{\phi^*E_-\otimes E} - \bar{f}_\omega(\sigma),$$

where the notation $\bar{\eta}_V = \frac{\eta(D_V)+\dim\ker D_V}{2} \pmod{\mathbb{Z}}$ is the reduced eta invariant [2, 3] of Atiyah–Patodi–Singer of the Dirac operator associated to the Spin^c-structure of M with coefficients in the vector bundle V [1] and

$$\bar{f}_\omega(M,E,\phi) = \left[\int_M \phi^*(\omega)\mathrm{Ch}(E)\mathrm{Td}(M)\right].$$

Let us check that $\alpha(\mathcal{E})(\partial(\sigma,\tau)) = 0$ in $\mathbb{R}/\mathbb{Z}$ for any K-chain σ over X and any K-chain τ over Y. Recall that $\partial(\sigma,\tau) = (\partial\sigma + \rho^*\tau, -\partial\tau)$. Furthermore, the invariant $\bar{\eta}$ and $\bar{f}_\omega$ defines K-cochains over X [8]. We have then

$$\alpha(\mathcal{E})(\partial(\sigma,\tau)) = \alpha(\mathcal{E})(\partial\sigma, -\partial\tau) + \alpha(\mathcal{E})(\rho^*\tau, 0).$$

The index theorem of APS (see [1, 2, 3]) implies that

$$\bar{\eta}_{(\phi^*E_+\otimes E)|\partial M} - \bar{\eta}_{(\phi^*E_-\otimes E)|\partial M} - \bar{f}_{d\omega}(\sigma) = \mathrm{ind}(D_+ \otimes \phi^*E_+ \otimes E) - \mathrm{ind}(D_+ \otimes \phi^*E_- \otimes E),$$

is entire, where $\mathrm{ind}(D_+ \otimes \phi^* E_\pm \otimes E)$ is the index of the Dirac type operator associated to the Spin^c-structure of M with coefficients in the bundle $\phi^* E_\pm \otimes E$. On the other hand, we have

$$\alpha(\mathcal{E})(\rho^*\tau, 0)) = \alpha(\rho^*\mathcal{E})(0, \tau) = 0.$$

This construction defines a homomorphism $\alpha : \bar{K}^{-1}(\rho, \mathbb{R}/\mathbb{Z}) \to \mathrm{Hom}(K_{-1}(\rho), \mathbb{R}/\mathbb{Z})$ which is a split of ς_1^*. In fact, let us consider the following commutative diagram

$$\begin{array}{ccc} \bar{K}^{-1}(\rho, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\alpha} & \mathrm{Hom}(K_{-1}(\rho), \mathbb{R}/\mathbb{Z}) \\ \Big\downarrow i^* & & \Big\downarrow \varsigma_1^* \\ K^{-1}(X, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\alpha_X} & \mathrm{Hom}(K_{-1}(X), \mathbb{R}/\mathbb{Z}) \end{array}$$

where i^* is the embedding of $\bar{K}^{-1}(\rho, \mathbb{R}/\mathbb{Z})$ in $K^{-1}(X, \mathbb{R}/\mathbb{Z})$ and α_X is the restriction of α to $\mathcal{C}_*(X) \times \{0\}$ which is an isomorphism [14]. For any $\mathcal{E} \in \bar{K}^{-1}(\rho, \mathbb{R}/\mathbb{Z})$, we have $\varsigma_1^*(\alpha(\mathcal{E})) = i^*(\mathcal{E}) = \mathcal{E}$. ∎

Theorem 1. *The following sequence is exact:*

$$0 \to K^{-1}(\rho, \mathbb{R}/\mathbb{Z}) \hookrightarrow \hat{K}^{-1}(\rho) \xrightarrow{\delta_1} \Omega_0^{\mathrm{even}}(\rho) \to 0.$$

Proof. From Proposition 1, $K^{-1}(\rho, \mathbb{R}/\mathbb{Z})$ is isomorphic to $\mathrm{Hom}(K_{-1}(\rho), \mathbb{R}/\mathbb{Z})$ which obviously injects in $\hat{K}^{-1}(\rho)$ with trivial δ_1. It is clear that a relative differential K-character f with $\delta_1(f) = 0$, induces a homomorphism from $K_{-1}(\rho)$ to $\mathbb{R}/\mathbb{Z}$. Hence the sequence is exact at $K^{-1}(\rho)$. It remains to show the surjectivity of δ_1.

Let $(\omega, \theta) \in \Omega_0^{\mathrm{even}}(\rho)$ and $f_{\omega,\theta} : \mathcal{L}_*(\rho) \to \mathbb{R}/\mathbb{Z}$ defined by

$$f_{\omega,\theta}(\sigma, \tau) = \overline{f_\omega(\sigma)} + \overline{f_\theta(\tau)}.$$

The map $f_{\omega,\theta}$ is trivial on $\mathcal{C}_{-1}(\rho)$. Therefore, we define an element $\chi \in \mathrm{Hom}(\mathcal{B}_{-1}(\rho), \mathbb{R}/\mathbb{Z})$ by setting

$$\chi(\partial(\sigma, \tau)) = f_{\omega,\theta}(\sigma, \tau),$$

where $\mathcal{B}_{-1}(\rho)$ is the image of the boundary map $\partial : \mathcal{L}_0(\rho) \to \mathcal{C}_{-1}(\rho)$.

Since $\mathbb{R}/\mathbb{Z}$ is divisible, χ can be extended to a relative differential K-character $\overline{\chi} : \mathcal{C}_{-1}(\rho) \to \mathbb{R}/\mathbb{Z}$ with $\delta_1(\overline{\chi}) = (\omega, \theta)$. ∎

3.2 Application

Let G be an almost connected Lie group and M be a smooth compact manifold. Let $\pi : Y \to M$ be a compact principal G-bundle with connection ∇. We denote by $I^*(G)$ the ring of symmetric multilinear real functions on the Lie algebra of G which are invariant under the adjoint action of G [11]. Let Ω be the curvature of ∇. For any $P \in I^*(G)$, there is a well defined closed form $P(\Omega)$ on M. The pullback $\pi^* P(\Omega)$ is an exact form on Y [11]. For $P \in I^*(G)$, let $TP(\nabla)$ be such that $\pi^* P(\Omega) = dTP(\nabla)$. The form $\omega = (\pi^* P(\Omega), dTP(\nabla))$ is a closed form in the complex $(\Omega^*(\pi), \delta)$. The relative differential K-character f_ω has a trivial δ_1 and defines consequently an element of the group $K^{-1}(\pi, \mathbb{R}/\mathbb{Z})$. This gives an additive map from $I^*(G)$ to $K^{-1}(\pi, \mathbb{R}/\mathbb{Z})$ which can be looked as a home of secondary geometric invariants of the principal G-bundle with connection (M, Y, ∇) analogous to the Chern–Weil theory.

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